Math2050b HW6

1. Suppose $I = (a, a+1)$. If $a = 0$, then it is trivial. If $a \in \mathbb{Z}$, suppose $(m,n) = 1$, then $(m - an, n) = 1$. It reduces back to the case of $a = 0$. If $a \in \mathbb{Q}$, say $a = \frac{\alpha}{\beta}$ with $\alpha, \beta \in \mathbb{Z}$. Then it is easy to see that B_n in this case has at most n elements. As rational number is dense, and each B_n is finite. the case when $a \notin \mathbb{Q}$ follows.

2. Let $I = (a,b)$ be the interval with positive length. Suppose f is bounded on I. That is $\exists M > 0$ such that for all $x \in I$, $|f(x)| \leq M$. Let $\frac{m}{n} \in I$, then $f(x) = n \leq M$. Hence, for sufficiently large k, $B_k = \emptyset$. But $\cup_{n=1}^{\infty} B_n = \mathbb{Q} \cap I$ which is impossible.

3. Suppose $\lim_{x \to x_0} f(x)$ does not exist. By cauchy criterion, there is $\epsilon_0 > 0$ such that for all $\delta > 0$, we can find $x, y \in V_\delta(x_0) \setminus \{x_0\}$ such that
$$|f(x) - f(y)| \geq \epsilon_0.$$
We can find a sequence x_n, y_n by considering $\delta = \frac{1}{n}$ so that for all n,
$$|f(x_n) - f(y_n)| \geq \epsilon_0 > 0.$$
If f is bounded, by BW, we can extract subsequence such that $f(x_{n_k})$ and $f(y_{n_k})$ is convergent with distinct limit due to the above inequality.